Name: ───────────────────────────

ID number: ──────────────────────────

Instructions:

1. You have 50 minutes to complete this exam.

2. There are 10 problems on this exam. Eight are multiple choice and two of them are free-response problems.

3. **Circle** one **and only one** option for each multiple-choice problem. No partial credit will be given for multiple-choice problems.

4. Show **all** relevant work on free-response problems. Partial credit will be given for clear steps leading to solutions. **Little to no credit will be given for little to no work.**

5. No books, notes, or calculators are allowed.

6. Please turn off your cell phone.

I agree to abide by the instructions above:

Signature: ──────────────────────────

1. (10 points) Circle either True (**T**) or False (**F**) for the following statements:

T / F : If an $n \times n$ matrix is invertible, then 0 is not one of its eigenvalues.

T / F : Let A be a square matrix. If A is diagonalizable, then A^k is diagonalizable for any integer $k \geq 1$.

T / F : The map $T : M_{2\times 2} \to \mathbb{R}^1$ given by $T(A) = \det A$ is a linear transformation.

T / F : If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

T / F : If an $n \times n$ matrix is diagonalizable, then it has n distinct eigenvalues.

2. (8 points) Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 & 5 \\ 0 & 1 & 4 & -6 & 7 \\ 0 & 0 & -2 & 1 & 4 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Which of the following statements are true for the matrix A?

(i) A has four distinct eigenvalues.

(ii) 1 is an eigenvalue of A of multiplicity 2.

(iii) -2 is an eigenvalue whose associated eigenspace has dimension 2.

A. (i) only

B. (ii) only

C. (ii) and (iii) only

D. (i) and (ii) only

E. (i) and (iii) only

F. (i), (ii), and (iii)

3. (9 points) Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the linear transformation given by $T(p(t)) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}$, where $\mathbb{P}_2$ is the vector space of polynomials with real coefficients of degree at most 2. Find the dimension of the kernel of T and the dimension of the range of T.

    A. $\dim(\ker(T)) = 2$, $\dim(\mathrm{range}(T)) = 1$

    B. $\dim(\ker(T)) = 2$, $\dim(\mathrm{range}(T)) = 2$

    C. $\dim(\ker(T)) = 1$, $\dim(\mathrm{range}(T)) = 2$

    D. $\dim(\ker(T)) = 3$, $\dim(\mathrm{range}(T)) = 1$

    E. $\dim(\ker(T)) = 2$, $\dim(\mathrm{range}(T)) = 0$

    F. $\dim(\ker(T)) = 0$, $\dim(\mathrm{range}(T)) = 1$

4. (9 points) Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$. If $A = PDP^{-1}$ for invertible P and diagonal matrix D, which of the following could be D?

    A. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

    B. $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$

    C. $\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$

    D. $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$

    E. $\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

    F. $\begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$

5. (9 points) Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation given by $T(p(t)) = p'(t)+p(t)$. Which of the following is the matrix representation of T with respect to the standard basis $\beta = \{1, t, t^2\}$?

A. $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

F. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

6. (9 points) Which **one** of the following statements is **false** for the matrix $\begin{bmatrix} 11 & 10 \\ 1 & 2 \end{bmatrix}$?

    A. A has exactly 2 eigenvalues

    B. There exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = -\mathbf{x}$

    C. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector for A

    D. There exist 2 linearly independent eigenvectors for A

    E. $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ is an eigenvector for A

    F. There exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = 12\mathbf{x}$

7. (9 points) Which **one** of the following sets is **not** a subspace of $M_{2\times2}$, where $M_{2\times2}$ is the vector space of 2×2 matrices with real entries?

    A. the set of all matrices of the form

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix},$$

    where a, b and d are real numbers.

    B. $\text{Span}\left\{ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

    C. $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

    D. The set of 2×2 matrices with real entries and determinant 0

    E. The set of all 2×2 diagonal matrices with real entries

    F. The set of all 2×2 lower triangular matrices with real entries

8. (9 points) Given that $\lambda = 5$ is an eigenvalue of multiplicity 2 for

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix},$$

which of the following is a basis of eigenvectors for the eigenspace associated to $\lambda = 5$?

A. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

F. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

9. (14 points) Suppose $x'(t) = A \cdot x(t)$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, and $x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$

(a) Find a fundamental set of solutions for the system of differential equations.

(b) Find the unique solution $x(t)$ with $x(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

10. (14 points) Diagonalize the following matrix over the complex numbers if possible. In other words find diagonal matrix D and invertible matrix P such that $A = PDP^{-1}$. **You do not need to compute P^{-1}!**

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$